

p -central action on groups

Yassine Guerboussa^a

^aUniversity Kasdi Merbah Ouargla, Ouargla, Algeria
Email: yassine_guer@hotmail.fr

Abstract

Let G be a finite p -group acted on faithfully by a group A . We prove that if A fixes every element of order dividing p (4 if $p = 2$) in a specified subgroup of G , then both A and $[G, A]$ behave regularly, that is the elements of order dividing any power p^i in each one of them form a subgroup; moreover A and $[G, A]$ have the same exponent, and they are nilpotent of class bounded in terms of p and the exponent of A . This leads in particular to a solution of a problem posed by Y. Berkovich. In another direction we discuss some aspects of the influence of a p -group P on the structure of a finite group which contains P as a Sylow subgroup, under assumptions like every element of order p (4 if $p = 2$) in a given term of the lower central series of P lies in the center of P .

Keywords: automorphisms, finite p -groups

1. Introduction

Let G be a finite group acted on by a group A . It is convenient to say that A acts p -centrally on G if A fixes every element of order dividing p (4 if $p = 2$) in G .

For a positive integer k , the left normed commutator $[x_1, x_2, \dots, x_k]$ in k elements of an ambient group, can be defined by induction, $[x_1] = x_1$ and $[x_1, x_2, \dots, x_k] = [[x_1, x_2, \dots, x_{k-1}], x_k]$.

We define $\gamma_k(G, A)$ to be the subgroup of G generated by all the left normed commutators $[x_1, x_2, \dots, x_n]$, $n \geq k$, where the x_i 's lie in $G \cup A$ in such a way that $x_1 \in G$, and at least $k - 1$ of them lie in A . Note that if one takes the natural action of G on itself, $\gamma_k(G, G)$ coincides with $\gamma_k(G)$ the k th term of the lower central series of G . Moreover we have $\gamma_k(G, A)$ is an A -invariant normal subgroup of G ; this fact will be used freely below.

In [7], M. Isaacs proved that if A is cyclic and acts p -centrally on $[G, A]$, for all the primes p dividing $|G|$, then there is a severe restriction on the structure of $[G, A]$ in terms of n the order of A ; for instance $[G, A]$ is nilpotent of class bounded by n , and has exponent dividing n . The first purpose of this paper is to show in one hand that an analogue of Isaacs' result holds under the weaker condition that A is a group of automorphisms of G acting p -centrally on $\gamma_p(G, A)$, and on the other hand to show that such a severe restriction applies also on A .

Theorem 1.1. *Let G be a finite p -group and A be a group of automorphisms of G , such that A acts p -centrally on $\gamma_p(G, A)$. Then for all positive integer i ,*

(i) *the elements of $[G, A]$ of order dividing p^i form a subgroup;*

Preprint submitted to Elsevier

May 5, 2014

- (ii) the elements of A of order dividing p^i form a subgroup;
- (iii) $\exp([G, A]) = \exp(A)$;
- (iv) the nilpotency class of both A and $[G, A]$ does not exceed $n + p - 2$, where $p^n = \exp(A)$.

This result is the best possible as shows the following example :

Let E be an elementary abelian p -group of rank $p + 1$, and let A be the automorphism group of E generated by the matrix

$$\sigma = \begin{pmatrix} 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 & \\ & & & \ddots & 1 \\ & & & & 1 \end{pmatrix}$$

Clearly, $\gamma_p(E, A) = [E, {}_pA]$, and A acts p -centrally on it. We have $\exp[E, A] = p$, however it is easy to see that for any positive integer n , we have (with the convention $\binom{n}{i} = 0$ for $i \geq n + 1$)

$$\sigma^n = \begin{pmatrix} 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{p} \\ & 1 & \binom{n}{1} & \ddots & \vdots \\ & & \ddots & \ddots & \binom{n}{2} \\ & & & 1 & \binom{n}{1} \\ & & & & 1 \end{pmatrix}$$

Therefore $\sigma^p \neq 1$, that is $\exp(A) \geq p^2$.

The following is an immediate consequence of Theorem 1.1.

Corollary 1.2. *Let G be a finite p -group which acts p -centrally on $\gamma_p(G)$. Then G' and $G/Z(G)$ have the same exponent.*

Note that a particular version of Corollary 1.2 was proved by T. Laffey in [11], where he established it under the condition $\Omega(G) \leq Z(G)$. M. Y. Xu generalized Laffey's result to the case where G acts p -centrally on $\gamma_{p-1}(G)$, with p odd (see [12, Corollary 4]). The result is also known for a special class of p -groups of class $\leq p$, more precisely for p -groups of maximal class and order $\leq p^{p+1}$. Moreover our proof implies, in fact, that $[\Omega_k(G/Z(G)), G] = \Omega_k(G')$, for any positive integer k .

As another application of Theorem 1.1, we solve the following problem posed by Y. Berkovich.

Problem 1891 [3]. Do there exist a prime p and a group G of order p^p and exponent p such that p^2 divides $\exp(\text{Aut}(G))$?

Let G be finite group of order $\leq p^p$ and exponent p , and let A be a p -Sylow of $\text{Aut}(G)$. According to Lemma 2.1 (iii) below, $\gamma_{i+1}(G, A) < \gamma_i(G, A)$, unless $\gamma_i(G, A) = 1$. Therefore $|\gamma_i(G, A)| \leq p^{p-i+1}$, so that $|\gamma_p(G, A)| \leq p$. It follows that A acts p -centrally on $\gamma_p(G, A)$, hence Theorem 1.1 yields

Corollary 1.3. *Let G be a non-cyclic group of order $\leq p^p$ and exponent p . Then a p -Sylow of $\text{Aut}(G)$ has exponent p .*

A finite group G which acts (by conjugation) p -centrally on itself is termed p -central (this terminology is due to A. Mann, see [5]). The p -central p -groups have many nice properties which qualify them to be dual to the powerful p -groups, we refer the reader to the introduction of [5] for some basic facts on p -central p -groups.

J. González-Sánchez and T. Weigel introduced in [5] a class of groups that are more general than the p -central ones. They called a group G p -central of height k , if every element of order p lies in the k th term of the upper central series of G . The first main result in their paper is

Theorem 1.4 (González-Sánchez and T. Weigel). *Let G be a finite p -central group of height $k \geq 1$, with p odd. Then G has a normal p -complement.*

A natural variant of a p -central group of height k , is a group that acts p -centrally on the k th term of its lower central series. For $k = 1$ and p odd, the two definitions coincide. A remarkable work in this context, which is not followed up extensively, was done by Ming-Yao Xu in [13]. He proved that a finite p -group (p odd) satisfying $\Omega_1(\gamma_{p-1}(G)) \leq Z(G)$ should behave regularly: the exponent of $\Omega_n(G)$ does not exceed p^n , and moreover $|G : G^{p^n}| \leq |\Omega_n(G)|$, for all positive integer n .

The next result deals with the analogue of Theorem 1.4 for this dual class. Note that this generalizes Lemma B in [7], with only a slight more effort. The proof follows easily from Frobenius' normal p -complement theorem (see [6, Theorem 4.5, p 253]).

Theorem 1.5. *Let G be a finite group which acts p -centrally on $\gamma_i(G)$ for some positive integer i . Then G has a normal p -complement.*

Let us note that the original proof of Theorem 1.4 is based on Quillen stratification (see [5, Theorem 3.1]), as well as Quillen's p -nilpotency criterion (see [5, Theorem 3.3]). We will give below a more elementary proof of it, which is based on Theorem 1.5, and hence on the classic Frobenius' normal p -complement theorem. Note also that our proof covers the prime 2, however in that case we have to assume that G is 4-central of height $k \geq 1$.

Now we turn our attention to p -soluble groups. Assume that G is a finite p -soluble group, and that a p -Sylow of G is p -central (4-central if $p = 2$) of height k . In [5] it is proved that if $k \leq p - 2$ and $p \neq 2$, then the p -length of G is ≤ 1 . In a subsequent paper E. Khukhro (see [10]) generalized this result and showed that the p -length of G is bounded above by $2m + 1$, where m is the largest integer satisfying $p^m - p^{m-1} \leq k$. By means of a theorem of P. Hall and G. Higman (see [8, Theorem A (ii)]), such a result holds if one can find an appropriate bound on the exponent of a p -Sylow of $G/O_{p',p}(G)$. We have the following analogues of Khukhro's results.

Theorem 1.6. *Let G be a finite p -soluble group such that $O_{p'}(G) = 1$. If a p -Sylow P of G acts p -centrally on $\gamma_k(P)$, for some positive integer k ; then the exponent of a p -Sylow of $G/O_p(G)$ does not exceed p^m , where m is the largest integer satisfying $p^m - p^{m-1} \leq k$.*

This corollary follows easily from [8, Theorem A (ii)] for p odd, and from [4] for $p = 2$.

Corollary 1.7. *Let G be a finite p -soluble group such that a p -Sylow P of G acts p -centrally on $\gamma_k(P)$, for some positive integer k . Then the p -length of G is bounded above by $2m + 1$, where m is the largest integer satisfying $p^m - p^{m-1} \leq k$.*

The notation in this paper is standard. Note only that $\Omega(G)$ stands for $\Omega_1(G)$ if p is odd, and $\Omega_2(G)$ if $p = 2$; and $\Omega_{\{i\}}(G)$ denotes the set of all elements of G having order dividing p^i .

The basic results on regular p -groups can be found in [9, Kap III], as well as in [1]. We shall use them freely in the paper.

The remainder of the paper is divided into two sections. Section 2 is devoted to proving Theorem 1.1, and Section 3 to proving the remaining theorems stated in the introduction.

2. p -central action on p -groups

First, we collect some basic facts about the series $\gamma_i(G, A)$ defined in the introduction.

Lemma 2.1. *Let G be a finite group and A be a group acting on G . Then we have*

1. $[\gamma_i(G, A), \gamma_j(A)] \leq \gamma_{i+j}(G, A)$, for $i, j \geq 1$;
2. $\gamma_{i+1}(G, A) = \langle [\gamma_i(G, A), A_n G], n \geq 0 \rangle$;
3. *If A and G are finite p -groups, then $\gamma_{i+1}(G, A) < \gamma_i(G, A)$, unless $\gamma_i(G, A) = 1$.*

Proof. 1. We proceed by induction on j . Assume that $j = 1$, if c is a generator of $\gamma_i(G, A)$ and $a \in A$, then clearly $[c, a] \in \gamma_{i+1}(G, A)$. Since $\gamma_{i+1}(G, A)$ is normal in G and A -invariant, the claim follows for $j = 1$. Assume now that the result holds for j , it follows that $[\gamma_i(G, A), \gamma_j(A), A] \leq [\gamma_{i+j}(G, A), A] \leq \gamma_{i+j+1}(G, A)$, and $[A, \gamma_i(G, A), \gamma_j(A)] \leq [\gamma_{i+1}(G, A), \gamma_j(A)] \leq \gamma_{i+j+1}(G, A)$; the Three Subgroups Lemma yields $[\gamma_{j+1}(A), \gamma_i(G, A)] \leq \gamma_{i+j+1}(G, A)$.

2. It follows from the first property that all the subgroups $[\gamma_i(G, A), A_n G]$ lie in $\gamma_{i+1}(G, A)$. Conversely let $c = [x_1, x_2, \dots, x_k]$ be a generator of $\gamma_{i+1}(G, A)$, and let be $s = \sup\{i \mid x_i \in A\}$. We have $s \geq i + 1$, so $[x_1, x_2, \dots, x_{s-1}] \in \gamma_i(G, A)$, hence $c \in [\gamma_i(G, A), A_{k-s} G]$.

3. Assume for a contradiction that $\gamma_{i+1}(G, A) = \gamma_i(G, A) \neq 1$. Let be $N = \gamma_i(G, A) \cap Z(GA)$; we have $\gamma_i(G/N, A) = \gamma_i(G, A)/N$. By induction on the order of G , if $\gamma_i(G, A) \neq N$ then $\gamma_{i+1}(G, A)N < \gamma_i(G, A)$ which contradicts our assumption. Thus we have $\gamma_i(G, A) = N$. It follows that $[\gamma_i(G, A), A_n G] = 1$ for all $n \geq 0$, so by the second property we have $\gamma_{i+1}(G, A) = 1$, a contradiction. \square

In the same spirit, the Three Subgroups Lemma yields

Lemma 2.2. *Let G be a group and A be a group acting on G . Let k be an integer ≥ 2 , $H = [G, A]$, and assume that A acts p -centrally on $\gamma_k(G, A)$. Then $\Omega(\gamma_{k-1}(H)) \leq \Omega(\gamma_k(G, A)) \leq Z(H)$.*

Proof. We have $[\Omega(\gamma_k(G, A)), G, A] = [A, \Omega(\gamma_k(G, A)), G] = 1$. The Three Subgroups Lemma yields $[H, \Omega(\gamma_k(G, A))] = 1$. Since $\gamma_k(G, A) \leq \gamma_2(G, A) = H$, it follows that $\Omega(\gamma_k(G, A)) \leq Z(H)$.

Now we claim that $\gamma_{k-1}(H) \leq \gamma_k(G, A)$. For $k = 2$ this is trivial. Assume that this is proved for k , and put $K = \gamma_k(G, A)$. We have $[K, A, G] \leq \gamma_{k+1}(G, A)$, and $[G, K, A] \leq [K, A] \leq \gamma_{k+1}(G, A)$. It follows again from the Three Subgroups Lemma that $[H, K] \leq \gamma_{k+1}(G, A)$. By assumption $\gamma_{k-1}(H) \leq K$, therefore $\gamma_k(H) \leq \gamma_{k+1}(G, A)$. \square

In [12] Ming-Yao Xu showed that if a finite p -group G , p odd, satisfies $\Omega_1(\gamma_{p-1}(G)) \leq Z(G)$, then G is strongly semi- p -abelian, in other words G satisfies the property :

$$(xy^{-1})^{p^n} = 1 \Leftrightarrow x^{p^n} = y^{p^n} \text{ for any positive integer } n.$$

Such a group shares many properties with the regular ones (see [13]). For instance the exponent of $\Omega_n(G)$ does exceed p^n , and $|G : G^{p^n}| \leq |\Omega_n(G)|$. The above does not hold for $p = 2$ as shows the quaternion group. However if one requires that $\Omega_2(G) \leq Z(G)$ we cover the case of 2-central 2-groups which is covered for instance in [7, Corollary 2.2]. Hence we have

Lemma 2.3 (Ming-Yao Xu). *Let G be a finite p -group such that $\Omega(\gamma_{p-1}(G)) \leq Z(G)$. Then for any positive integer n , we have*

- (i) *the elements of G of order dividing p^n form a subgroup;*
- (ii) *for p odd, $|G : G^{p^n}| \leq |\Omega_n(G)|$.*

The above result combined with Lemma 2.2 yields

Corollary 2.4. *Let G be a finite p -group and A be a group acting on G , such that A acts p -centrally on $\gamma_p(G, A)$. Then for any positive integer n , and for $H = [G, A]$, we have*

- (i) $\exp(\Omega_n(H)) \leq p^n$;
- (ii) *for p odd, $|H : H^{p^n}| \leq |\Omega_n(H)|$.*

The following lemma generalizes a result of I. M. Isaacs (see [7, Theorem 2.1]).

Lemma 2.5. *Let G be a finite p -group and A be a group of order p acting on G , such that A acts p -centrally on $\gamma_p(G, A)$. Then $[G, A]$ has exponent at most p .*

Proof. Assume first that p is odd. By induction on $|G|$ we may assume that the result holds for any smaller p -group. We have $[G, A] < G$, thus by induction $[G, A, A]$ has at most exponent p . This yields $[G, A, A] \leq \Omega_1(H)$; since $\Omega_1(H)$ is normal in G , Lemma 2.1(ii) and Corollary 2.4 (i) imply that $\gamma_3(G, A)$ has exponent $\leq p$. Our condition on the action of A on G implies in particular that $[\gamma_p(G, A), A] = 1$, and again Lemma 2.1(ii) yields $\gamma_{p+1}(G, A) = 1$.

Now let K denote the semidirect product $[G, A]A$. We claim that $\gamma_p(K) = 1$. This follows at once if one proves that $\gamma_{i-1}(K) \leq \gamma_i(G, A)$ for all $i \geq 3$. Assume that $i = 3$. It follows easily from the Three Subgroup Lemma that $[[G, A], [G, A]] \leq \gamma_3(G, A)$, and clearly we have $[[G, A], A] \leq \gamma_3(G, A)$. This amounts to saying that $[G, A]/\gamma_3(G, A)$ lies in the center of $K/\gamma_3(G, A)$, but since $K/[G, A] \cong A$ is cyclic, we have $K/\gamma_3(G, A)$ is abelian. Hence $\gamma_2(K) \leq \gamma_3(G, A)$. Now by induction we may assume that $\gamma_{i-1}(K) \leq \gamma_i(G, A)$. We have $[\gamma_i(G, A), A] \leq \gamma_{i+1}(G, A)$, and the Three Subgroups Lemma implies that $[\gamma_i(G, A), [G, A]] \leq \gamma_{i+1}(G, A)$. Thus $[\gamma_{i-1}(K), K] \leq [\gamma_i(G, A), K] \leq \gamma_{i+1}(G, A)$.

As $\gamma_p(K) = 1$, it follows that K is regular; moreover as $K' \leq \gamma_3(G, A)$, we have $\exp(K') \leq p$. Thus $(xy)^p = x^p y^p$ for all $x, y \in K$; in other words K is p -abelian.

Let be $g \in G$, $\sigma \in A$ and set $x = [g, \sigma]$. we have

$$g^{-1} g^{\sigma^p} = x^{1+\sigma+\dots+\sigma^{p-1}} = 1$$

Since K is p -abelian, we have

$$x^{1+\sigma+\dots+\sigma^{p-1}} = (x\sigma^{-1})^p \sigma^p = x^p \sigma^{-p} \sigma^p = x^p$$

This shows that $[G, A]$ is generated by elements of order not exceeding p ; as K is regular it follows that the exponent of $[G, A]$ is at most p .

For $p = 2$, we may assume that $[G, A, A]$ has exponent ≤ 2 . Under the above notation let $y = [x, \sigma]$. We have $1 = g^{-1}g^{\sigma^2} = x^2y$ and $y^2 = 1$, hence $x^4 = 1$. As A fixes every element of order 4 in $[G, A]$, we have $y = [x, \sigma] = 1$. Thus $x^2 = 1$ and on the other hand $\gamma_3(G, A) = 1$. Now as $[[G, A], [G, A]] \leq \gamma_3(G, A) = 1$, it follows that $[G, A]$ is abelian, and it is generated by elements of order ≤ 2 , the result follows. \square

The following lemma is useful for induction.

Lemma 2.6. *Let G be a finite p -group and A be a p -group acting on G . If A acts p -centrally on $\gamma_k(G, A)$, for some $1 \leq k \leq p$, then the same holds if we replace G by $G/\Omega_i(H)$ for any positive integer i , where H denotes $[G, A]$.*

Proof. Assume first that $i = 1$ and let us denote $G/\Omega_1(H)$ by \overline{G} . We have $\gamma_k(\overline{G}, A) = \overline{\gamma_k(G, A)}$. Let $\overline{x} \in \gamma_k(\overline{G}, A)$ be an element of order $\leq p$ (≤ 4 if $p = 2$), so we can assume that $x \in \gamma_k(G, A)$. As $\gamma_p(G, A) \leq \gamma_k(G, A)$, it follows from Corollary 2.4 (i) that $x^p \in \Omega(\gamma_k(G, A))$. Therefore $[x^p, A] = 1$, and by Lemma 2.2 x^p lies in the center of H . Thus x induces on $K = HA$ (by conjugation) an automorphism of order $\leq p$.

It follows easily that $\gamma_k(K, \langle x \rangle) \leq \gamma_{k-1}(H, \langle x \rangle) \leq \gamma_{k-1}(H)$. By Lemma 2.2, $\gamma_{k-1}(H) \leq Z(H)$, thus the inner automorphism induced by x on K acts p -centrally on $\gamma_k(K, \langle x \rangle)$, so it acts p -centrally on $\gamma_p(K, \langle x \rangle)$. It follows at once from Lemma 2.5 that $[x, A]^p = 1$, that is $[\overline{x}, A] = \overline{1}$.

Now by induction we may assume the result is true for $G/\Omega_i(H)$. Since $[G/\Omega_i(H), A] = H/\Omega_i(H)$, Corollary 2.4 (i) implies that $\Omega_1([G/\Omega_i(H), A]) = \Omega_{i+1}(H)/\Omega_i(H)$, so by the first step the result holds for $G/\Omega_{i+1}(H)$. \square

It follows from the lemma above that

Lemma 2.7. *Let L denote $\gamma_k(G, A)$. Under the assumptions of Lemma 2.6, we have A acts p -centrally on $L/\Omega_i(L)$, and $[A, \Omega_i(L)] \leq \Omega_{i-1}(L)$, for all positive integer i .*

Proof. We have $\gamma_k(G/\Omega_i(H), A) = \gamma_k(G, A)\Omega_i(H)/\Omega_i(H) = L\Omega_i(H)/\Omega_i(H)$. Also $L\Omega_i(H)/\Omega_i(H)$ and $L/L \cap \Omega_i(H)$ are canonically isomorphic as A -groups. It follows from Corollary 2.4 (i) that $L \cap \Omega_i(H) = \Omega_i(L)$. Now Lemma 2.6 implies that A acts p -centrally on $L/\Omega_i(L)$. Also Corollary 2.4 (i) implies that $\Omega_1(L/\Omega_{i-1}(L)) = \Omega_i(L)/\Omega_{i-1}(L)$, thus $[A, \Omega_i(L)] \leq \Omega_{i-1}(L)$. \square

In Lemma 2.6 as well as Lemma 2.7 we assumed that A is a p -group. The following result shows that this assumption can be dropped if one assumes that A acts faithfully on G . The result in a seemingly weaker form is classic (see [9, Satz IV.5.12]).

Proposition 2.8. *Let G be a finite p -group and $A \leq \text{Aut}(G)$. If A acts p -centrally on $\gamma_i(G, A)$ for some $i \geq 1$, then A is a p -group.*

Proof. Let Q be a q -Sylow of A , $q \neq p$. By [6, Theorem 3.6, p 181], $[G, Q, Q] = [G, Q]$, so $[G, Q] = [G, Q]$. As $[G, Q] \leq \gamma_{i+1}(G, Q)$, it follows that Q acts p -centrally on $[G, Q]$, and from [7, Lemma 4.1] it follows that Q is a p -group. Thus $Q = 1$. \square

The collection of the lemmas above yields the following key result.

Proposition 2.9. *Let G be a finite p -group and A be a group of automorphisms of G , such that A acts p -centrally on $\gamma_p(G, A)$. Then an element $\sigma \in A$ satisfies $\sigma^{p^n} = 1$ if and only if σ acts trivially on $G/\Omega_n(H)$, where $H = [G, A]$.*

Proof. For $n = 1$, let be $\sigma \in A$ such that $\sigma^p = 1$. It follows immediately from Lemma 2.5 that $[G, \sigma]$ has exponent $\leq p$, that is σ acts trivially on $G/\Omega_1(H)$.

Conversely, assume that $[g, \sigma] \in \Omega_1(H)$, for all $g \in G$. Let be $K = [G, \langle \sigma \rangle] \langle \sigma \rangle$; as we seen in the proof of Lemma 2.5 $\gamma_p(K) \leq \gamma_{p+1}(G, \langle \sigma \rangle)$, and as $\gamma_p(G, \langle \sigma \rangle)$ has exponent p , it follows that $\gamma_{p+1}(G, \langle \sigma \rangle) = 1$. Therefore K has class $\leq p - 1$, so it is regular, and since $K' \leq [G, \langle \sigma \rangle]$ has exponent p , K is p -abelian. Now let be $g \in G$ and $x = [g, \sigma]$. We have

$$g^{-1}g^{\sigma^p} = x^{1+\sigma+\dots+\sigma^{p-1}} = (x\sigma^{-1})^p\sigma^p = x^p\sigma^{-p}\sigma^p = 1$$

thus σ has order $\leq p$.

Now we proceed by induction on n . If $\sigma^{p^{n+1}} = 1$, then σ^p has order at most p^n , so by induction $g^{-1}\sigma^p(g) \in \Omega_n(H)$ for all $g \in G$. Therefore σ acts on $G/\Omega_n(H)$ as an automorphism of order $\leq p$. By Lemma 2.6, σ acts p -centrally on $\gamma_p(G/\Omega_n(H), A)$, therefore $[G/\Omega_n(H), \langle \sigma \rangle]$ has exponent $\leq p$ by Lemma 2.5. Thus we have $[g, \sigma]^p \in \Omega_n(H)$ for all $g \in G$, and by Corollary 2.4 (i), $[g, \sigma]^{p^{n+1}} = 1$ for all $g \in G$.

Similarly, assume that $[G, \sigma] \leq \Omega_{n+1}(H)$. By Corollary 2.4 (i), $\exp(\Omega_{n+1}(H)) \leq p^{n+1}$, hence we have $[g, \sigma]^{p^{n+1}} = 1$, for all $g \in G$. Therefore the automorphism σ' induced by σ on $G/\Omega_n(H)$ satisfies $[g\Omega_n(H), \sigma']^p = 1$. We deduce from the first step that σ' has order $\leq p$. Thus $g^{-1}\sigma^p(g) \in \Omega_n(H)$ for all $g \in G$. By induction the order of σ^p is at most p^n , the result follows. \square

Now we can prove our first main theorem.

Proof of Theorem 1.1. (i) this is Corollary 2.4 (i).

(ii) Proposition 2.9 implies that $\Omega_{[i]}(A) = C_A(G/\Omega_i(H))$, thus $\Omega_{[i]}(A)$ is a subgroup of A .

(iii) Let be $\exp(H) = p^n$ and $\exp(A) = p^m$. We have $\Omega_n(A) = C_A(G/\Omega_n(H)) = C_A(G/H) = A$, it follows from (ii) that $p^m \leq p^n$. Again we have $A = \Omega_m(A) = C_A(G/\Omega_m(H))$. Therefore $[G, A] = H = \Omega_m(H)$. By Corollary 2.4 (i), $p^n \leq p^m$.

(iv) By Lemma 2.2, H acts p -centrally on $\gamma_{p-1}(H)$. Lemma 2.7 yields $[\Omega_i(\gamma_{p-1}(H)), H] \leq \Omega_{i-1}(\gamma_{p-1}(H))$, for all $i \geq 1$. Therefore

$$1 \leq \Omega_1(\gamma_{p-1}(H)) \leq \Omega_2(\gamma_{p-1}(H)) \leq \dots \leq \Omega_n(\gamma_{p-1}(H)) = \gamma_{p-1}(H) \leq \gamma_{p-2}(H) \leq \dots \leq H$$

is a central series of H . This proves that H is nilpotent of class $\leq n + p - 2$.

Similarly, let be $L = \gamma_p(G, A)$; Lemma 2.7 yields $[A, \Omega_i(L)] \leq \Omega_{i-1}(L)$, for all positive integer i . Therefore A stabilizes the normal series

$$1 \leq \Omega_1(L) \leq \Omega_2(L) \leq \dots \leq \Omega_n(L) = \gamma_p(G, A) \leq \gamma_{p-1}(G, A) \leq \dots \leq \gamma_2(G, A) \leq G$$

It follows at once from the well known result of Kaloujnine (see [9, Satz III.2.9]) that A is nilpotent of class $\leq n + p - 2$. \square

3. p -central action, p -nilpotency and p -solubility length

In the following proofs we need only Proposition 2.8, which is by the way independent from the material developed in the previous section.

Proof of Theorem 1.5. Let P be a non-trivial p -subgroup of G . We have $A = N_G(P)/C_G(P)$ acts faithfully on P , and p -centrally on $\gamma_i(P, A)$. By Proposition 2.8, A is a p -groups. The result follows now from Frobenius' criterion of p -nilpotency (see [6, Theorem 4.5, p 253]). \square

Elementary proof of Theorem 1.4. Let be $C = C_G(\Omega(G))$. As $\Omega(C) \leq \Omega(G)$, we have $\Omega(C) \leq Z(C)$. Therefore C acts p -centrally on itself. It follows from Theorem 1.5 that C has a normal p -complement N , and since N is characteristic in C , N is also normal in G . Now we have only to show that N has a p -power index, indeed, As $\Omega(G) \leq Z_k(G)$, it follows that $\Omega(G)$ is nilpotent, so it is a p -group. On the other hand it follows by induction that $[\Omega(G)_i, G] \leq Z_{k-i}(G)$, for $i \leq k$. In particular $[\Omega(G)_k, G] = 1$. Therefore $A = G/C$ acts faithfully on $\Omega(G)$ and p -centrally on $\gamma_{k+1}(G, A)$. From Proposition 2.8, one deduces that $|G : C|$ is a p -power, and clearly $|C : N|$ is a p -power. \square

The remainder part is devoted to proving Theorem 1.6. The proof is inspired by that of Khukhro in [10]. Let us quote some celebrated results that we need in the sequel.

Let P be a finite p -group. Recall that a Thompson critical subgroup of P is a characteristic subgroup C having the following properties:

1. $[C, P] \leq Z(C)$;
2. $C/Z(C)$ is elementary abelian;
3. C is self centralizing, that is $C_G(C) = Z(C)$;
4. any non trivial p' -automorphism of P acts non trivially on C .

A celebrated result of Thompson asserts that every finite p -group P has a Thompson critical subgroup C (see [6, Theorem 5.3.11]). By Proposition 2.8, any non trivial p' -automorphism of P acts non trivially on $K = \Omega(C)$, and the first two properties of C imply that K has exponent p (≤ 4 if $p = 2$). Hence we have the following well-known result.

Theorem 3.1 (Thompson). *Let P be a finite p -group. Then P has a subgroup K of exponent p (4 if $p = 2$) and class at most 2, such that any non trivial p' -automorphism of P acts non trivially on K .*

We need also the following weaker form of the Hall-Higman theorem (see [8, Theorem B]).

Theorem 3.2 (Hall-Higman). *Let V be a vector space over a field of characteristic p , and G be a p -soluble group of automorphisms of V such that $O_p(G) = 1$. If g is an element of G of order p^m , then the minimal polynomial of g is $(X - 1)^r$, where r satisfies $p^m - p^{m-1} \leq r \leq p^m$.*

Proof of Theorem 1.6. Let p^n be the exponent of a p -Sylow of $G/O_p(G) = \overline{G}$, and Q be a p -complement in $O_{pp'}(G)$.

Claim 1. There is $g \in G$ which normalizes Q such that $\overline{g} = g O_p(G)$ has order p^n .

Indeed, Let be $y \in G$ such that \bar{y} has order p^n . As any two p -complements in $O_{pp'}(G)$ are conjugate, Q and Q^y are conjugate in $O_{pp'}(G)$. Moreover since $O_{pp'}(G) = O_p(G)Q$, there is $x \in O_p(G)$ such that $Q^y = Q^x$. Now it suffices to take $g = yx^{-1}$.

Claim 2. \bar{g} acts faithfully on $\bar{Q} = QO_p(G)/O_p(G)$, so that $[Q, \bar{g}^{p^{n-1}}] \neq 1$.

Indeed, as $\bar{Q} = O_{p'}(\bar{G})$, it follows from [6, Theorem 6.3.2] that \bar{Q} is self centralizing. Thus $\langle \bar{g} \rangle \cap \bar{Q} = \bar{1}$.

Now let K be a subgroup of $O_p(G)$ as in Theorem 3.1. Consider a series of $O_p(G)$ -invariant subgroups

$$1 \leq K_1 \leq \dots \leq K_n = K$$

with elementary abelian sections, on which $O_p(G)$ acts trivially. Whence there is a well defined action (induced by conjugation) of the semi-direct product $A = Q\langle \bar{g} \rangle$ on each section K_{i+1}/K_i . Since $O_{p'}(G) = 1$, $O_p(G)$ is self centralizing by [6, Theorem 6.3.2]; therefore $[Q, g^{p^{n-1}}]$ is a non trivial p' -group of automorphisms of $O_p(G)$, and it acts non trivially on K by Theorem 3.1. Therefore $[Q, g^{p^{n-1}}]$ acts non trivially on some section $V = K_{i+1}/K_i$.

Claim 3. $O_p(A) = 1$, and \bar{g} acts faithfully on V .

Assume first that Claim 3 is true. It follows from Theorem 3.2 that $(\bar{g} - 1_V)^s \neq 0$, where $s = p^n - p^{n-1} - 1$. Thus for some $x \in K$, $[x_s, g] \neq 1$. On the other hand any p -Sylow P of G acts p -centrally on $\gamma_k(P)$, and since K has exponent p (4 if $p = 2$) we have $[x_k, g] = 1$. This implies that $k \geq s + 1 = p^n - p^{n-1}$.

It remains to prove Claim 3. As $\langle \bar{g} \rangle$ is a p -Sylow in A , if $O_p(A) \neq 1$, then $\bar{g}^{p^{n-1}} \in O_p(A)$. Hence $1 \neq [Q, \bar{g}^{p^{n-1}}] \leq O_p(A) \cap O_{p'}(A) = 1$, a contradiction. Also by definition of V , $[Q, \bar{g}^{p^{n-1}}]$ acts non trivially on it, so $\bar{g}^{p^{n-1}}$ acts non trivially on V . Thus \bar{g} acts faithfully on V . □

Acknowledgments

A. Mann took my attention to p -central p -groups, and provided me with a copy of [5]. An early discussion with M. I. Isaacs about his results in [7] was very helpful, and T. Laffey provided me with some of his very old papers. I'm really grateful to all of them.

References

- [1] Y. Berkovich, Groups of prime power order, vol. 1, Walter de Gruyter, 2008.
- [2] Y. Berkovich and Z. Janko, Groups of prime power order, vol. 2, Walter de Gruyter, 2008.
- [3] Y. Berkovich and Z. Janko, Groups of prime power order, vol. 3, Walter de Gruyter, 2011.
- [4] E. G. Bryukhanova, The 2-length and 2-period of a finite solvable group, *Algebra Logika* **18** (1979), 9-31; English transl., *Algebra Logic* **18** (1979), 5-20
- [5] J. González-Sánchez and T. S. Weigel, Finite p -central groups of height k , *Isr. J. Math.* **181** (2011), 125-143.
- [6] D. Gorenstein, Finite groups, Chelsea, New York, 1980.
- [7] I.M. Isaacs, Automorphisms fixing elements of prime order in finite groups, *Arch. Math.* **68** (1997), 359-366.
- [8] P. Hall and G. Higman, The p -length of a p -soluble group and reduction theorems for Burnside's problem, *Proc London Math. Soc.* (3) **6** (1956), 1-42.
- [9] B. Huppert, Endliche Gruppen. I. *Die Grundlehren der Mathematischen Wissenschaften*, Band 134. Springer-Verlag, Berlin, 1967.

- [10] E. Khukhro, On p -soluble groups with a generalized p -central or a powerful Sylow p -subgroup, *International Journal of Group Theory* **1** (2012), No. 2, 51-57
- [11] T. J. Laffey, Centralizers of Elementary Abelian Subgroups in Finite p -Groups, *Journal of Algebra* **51**, 88-96 (1978)
- [12] M.Y. Xu, A class of semi- p -abelian p -groups (in Chinese), *Kexue Tongbao* **26** (1981), 453-456. English translation in *Kexue Tongbao* (English Ed.) **27** (1982), 142-146
- [13] M.Y. Xu, The power structure of finite p -groups, *Bull. Aust. Math. Soc.* **36** (1987), no. 1, 1-10.